PROPERTIES OF FOURIER TRANSFORM

CTFT possesses a number of important properties. In this section we illustrate how they arise and their use in computation of Fourier Transforms.

Let $x(t)$ be the time-domain signal
$X(\omega)$ be its Fourier Transform

$$x(t) \xleftrightarrow{CTFT} X(\omega)$$

(1) LINEARITY OF FOURIER TRANSFORM

If $x_1(t) \longleftrightarrow X_1(\omega)$ and
$x_2(t) \longleftrightarrow X_2(\omega)$

Then,

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(\omega) + b X_2(\omega)$$

Proof:

$$FT\left[a x_1(t) + b x_2(t)\right] = \int_{-\infty}^{\infty} \left[a x_1(t) + b x_2(t)\right] e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} a x_1(t) e^{-j\omega t} \cdot dt + \int_{-\infty}^{\infty} b x_2(t) e^{-j\omega t} \cdot dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} \cdot dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} \cdot dt$$

$$= a X_1(\omega) + b X_2(\omega)$$

(2) TIME SHIFTING: If $x(t) \longleftrightarrow X(\omega)$

then $x(t-t_0) \longleftrightarrow e^{-j\omega t_0} \cdot X(\omega)$

Proof: $$FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} \cdot dt$$

Letting $(t-t_0) = \tau \Rightarrow t = (t_0+\tau)$

$dt = d\tau$

Also as $t \to -\infty$, $\tau \to -\infty$

and as $t \to \infty$, $\tau \to \infty$

$$\therefore FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(t_0+\tau)} \cdot d\tau = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} \cdot d\tau$$

$$FT[x(t-t_0)] = e^{-j\omega t_0} \cdot X(\omega).$$

Note: When a signal is shifted time, a phase shift is introduced into its FT. The magnitude of FT remains unaltered.

(3) FREQUENCY SHIFTING: If $x(t) \longleftrightarrow X(\omega)$

then $x(t) \cdot e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$

Proof: $$FT[x(t) e^{j\omega_0 t}] = \int_{-\infty}^{\infty} [x(t) \cdot e^{j\omega_0 t}] \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t} \cdot dt$$

$$= X(\omega-\omega_0)$$

$$FT[x(t) e^{j\omega_0 t}] = X(\omega-\omega_0).$$

Note: Multiplication of signal by a complex exponent in time domain corresponds to a frequency shift in the frequency domain

(4) TIME SCALING: If $x(t) \longleftrightarrow X(\omega)$

then $x(at) \longleftrightarrow \frac{1}{|a|} \cdot X\left(\frac{\omega}{a}\right)$

Proof: $FT[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t} \cdot dt$

(i) For +ve 'a'

Let $at = \tau \Rightarrow t = \tau/a$

$dt = \frac{1}{a} \cdot d\tau$

Also as $t \to -\infty$; $\tau \to -\infty$
and as $t \to \infty$; $\tau \to \infty$

$$FT[x(at)] = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\frac{\omega}{a}\cdot\tau} \cdot \frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\left(\frac{\omega}{a}\right)\cdot\tau} \cdot d\tau$$

$$FT[x(at)] = \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad \text{——— (1)}$$

(ii) In case 'a' is a -ve real value;

$FT[x(-at)] = \int_{-\infty}^{\infty} x(-at)\, e^{-j\omega t} \cdot dt$; Let $-at = \tau$

$dt = -\frac{1}{a} \cdot d\tau$

$$\therefore F[x(-at)] = \int_{\infty}^{-\infty} x(\tau)\, e^{-j\omega\left(\frac{-\tau}{a}\right)} \cdot \frac{d\tau}{-a}$$

as $t \to -\infty$, $\tau \to +\infty$
and as $t \to \infty$, $\tau \to -\infty$

$$= -\frac{1}{a}\int_{\infty}^{-\infty} x(\tau) \cdot e^{-j\left(\frac{-\omega}{a}\right)\tau} \cdot d\tau = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(-\frac{\omega}{a}\right)\cdot\tau} \cdot d\tau$$

$$F[x(-at)] = \frac{1}{a} X\left(\frac{-\omega}{a}\right) = \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad \text{——— (2)}$$

Combining the two cases, we have

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{|a|}\right)$$

Note: • time scaling by 'a' results in frequency scaling by $\frac{1}{|a|}$ and amplitude scaling by $\frac{1}{|a|}$.

• Compression of $x(t)$ to $x(at)$ leads to stretching of $X(\omega)$ by 'a' and an amplitude reduction by $|a|$

(5) TIME REVERSAL: If $x(t) \longleftrightarrow X(\omega)$

then $x(-t) \longleftrightarrow X(-\omega)$

Proof:

$$FT\left[x(-t)\right] = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t} \cdot dt$$

Let $-t = \tau$

$dt = -d\tau$

as $t \to -\infty$; $\tau \to \infty$

as $t \to \infty$; $\tau \to -\infty$

$$\therefore FT\left[x(-t)\right] = -\int_{\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau} \cdot d\tau = X(-\omega)$$

$$\boxed{x(-t) \longleftrightarrow X(-\omega)}$$

Note: • Reversing a signal in time-domain also reverses the Fourier Transform

(6) DIFFERENTIATION IN TIME DOMAIN: If $x(t) \longleftrightarrow X(\omega)$

then $\dfrac{dx(t)}{dt} \longleftrightarrow (j\omega)X(\omega)$

$\dfrac{d^2x(t)}{dt^2} \longleftrightarrow (j\omega)^2 X(\omega)$

proof: By definition, the inverse FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

Differentiating on both sides gives

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} (j\omega) \cdot d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega \cdot X(\omega)\right] e^{j\omega t} \cdot d\omega$$

$$\frac{dx(t)}{dt} = FT^{-1}\left[j\omega \cdot X(\omega)\right]$$

or $$FT\left[\frac{dx(t)}{dt}\right] = j\omega \cdot X(\omega)$$

Hence, $$\frac{dx(t)}{dt} \longleftrightarrow j\omega \cdot X(\omega) \quad //.$$

(7) DIFFERENTIATION IN FREQUENCY DOMAIN : If $x(t) \longleftrightarrow X(\omega)$

then $t \cdot x(t) \longleftrightarrow j\frac{dX(\omega)}{d\omega}$

Proof : By definition, $X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$

Differentiating w.r.to ω

$$\frac{d}{d\omega}\left[X(\omega)\right] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot (-jt) \cdot dt$$

$$= -j \int_{-\infty}^{\infty} \left[t \cdot x(t)\right] e^{-j\omega t} \cdot dt$$

$$\frac{d}{d\omega}\left[X(\omega)\right] = -j \; FT\left[t \cdot x(t)\right]$$

(or) $$FT\left[t \cdot x(t)\right] = j \frac{d}{d\omega} X(\omega)$$

$$t \cdot x(t) \longleftrightarrow j \frac{d}{d\omega}\left[X(\omega)\right]$$

Note : Multiplication of a signal $x(t)$ by t is equivalent to differentiation of its Fourier Transform.

(8) INTEGRATION PROPERTY : If $x(t) \longleftrightarrow X(\omega)$

$$\int_{-\infty}^{t} x(t) \cdot dt \longleftrightarrow \frac{X(\omega)}{j\omega} \; ; X(0) = 0$$

By definition $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$

Replacing t by a dummy variable 'τ'

$$x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega\tau} \cdot d\omega$$

Integrating both sides over $-\infty$ to t, we have

$$\int_{-\infty}^{t} x(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{t} e^{j\omega\tau} \cdot d\tau\right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\frac{e^{j\omega\tau}}{j\omega}\right]^{t} \cdot d\omega$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{X(\omega)}{j\omega}\right] e^{j\omega t}\cdot d\omega$$

$$= FT^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

or $$FT\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(\omega)}{j\omega}$$

$$\therefore \quad \int_{-\infty}^{t} x(\tau)\cdot d\tau \longleftrightarrow \frac{X(\omega)}{j\omega}$$

If $X(0) \neq 0$, then $x(t)$ is not an energy function and FT of $\int_{-\infty}^{t} x(\tau)\,d\tau$ includes an impulse i.e.,

$$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\cdot\delta(\omega)$$

Note :
- Integration of $x(t)$ in time domain is equivalent to Division of FT by $j\omega$
- Differentiation of $x(t)$ in time domain is equivalent to multiplication of FT by $j\omega$
- Differentiation of a function introduces a 90° phase shift in the spectrum and scales the magnitude of the spectrum in proportion to frequency
- Integration of a function introduces a -90° phase-shift in the spectrum and scales the magnitude of the spectrum inversely with the frequency.

(9) CONVOLUTION PROPERTY:

If $x_1(t) \longleftrightarrow X_1(\omega)$ and $x_2(t) \longleftrightarrow X_2(\omega)$

then $x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) X_2(\omega)$

proof:

$$FT\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} \left[x_1(t) * x_2(t)\right] \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) \cdot d\tau\right] e^{-j\omega t} \cdot dt$$

Changing the order of the integration, we have

$$FT\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t} \cdot dt\right] d\tau$$

This is $FT\left[x(t-\tau)\right] = e^{-j\omega\tau} X_2(\omega)$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot X_2(\omega)\, e^{-j\omega\tau} \cdot d\tau$$

$$= X_2(\omega) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} \cdot d\tau = X_2(\omega) \cdot X_1(\omega)$$

$$\therefore \quad FT\left[x_1(t) * x_2(t)\right] = X_1(\omega)\, X_2(\omega)$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega) \cdot X_2(\omega)$$

Note:
- Convolution of two signals in time domain is equivalent to multiplication of their spectra in frequency domain.
- Convolution in one domain corresponds to multiplication in the other domain.

(10) MULTIPLICATION (OR MODULATION) PROPERTY:

If $x_1(t) \longleftrightarrow X_1(\omega)$ & $x_2(t) \longleftrightarrow X_2(\omega)$

then $x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$

Proof:

$$FT\left[x_1(t) \cdot x_2(t)\right] = \int_{-\infty}^{\infty}\left[x_1(t) \cdot x_2(t)\right] e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty}\left[\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta) e^{j\theta t} \cdot d\omega\right) x_2(t)\right] e^{-j\omega t} \cdot dt$$

Interchanging the order of integration and noting that $X_1(\theta)$ does not depend on 't' yields

$$FT\left[x_1(t)\, x_2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)\left[\int_{-\infty}^{\infty} x_2(t) \cdot e^{j\theta t} \cdot e^{-j\omega t} \cdot dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)\left[\int_{-\infty}^{\infty} x_2(t) \cdot e^{-j(\omega-\theta)t} \cdot dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta) \cdot X_2(\omega-\theta)\, d\omega$$

$$FT\left[x_1(t)\, x_2(t)\right] = \frac{1}{2\pi}\left[X_1(\theta) * X_2(\theta)\right]$$

$$\therefore \quad x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi}\left[X_1(\theta) * X_2(\theta)\right]$$

Note:
- Multiplication of two signals in time domain is equivalent to convolution of their spectra in Frequency domain
- Multiplication in one domain is equivalent to convolution in the other domain.

(11) PARSEVAL'S RELATION: (OR) RAYLEIGH'S ENERGY THEOREM

If $x_1(t) \longleftrightarrow X_1(\omega)$ and $x_2(t) \longleftrightarrow X_2(\omega)$

then
$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t) \cdot dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega) \cdot d\omega$$

proof:

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, e^{j\omega t} \cdot d\omega\right] \cdot x_2^*(t) \cdot dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left[\int_{-\infty}^{\infty} x_2^*(t)\, e^{j\omega t} \cdot dt\right] \cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) \cdot \left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t} \cdot dt\right]^* \cdot d\omega$$

This is FT of $x_2(t) = X_2(\omega)$

$$\boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega) \cdot X_2^*(\omega) \cdot d\omega}$$

parseval's Identity: If $x_1(t) = x_2(t) = x(t)$, then

$$\int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, X^*(\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 \cdot dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Energy of signal

$$\boxed{E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega}$$

This is known as PARSEVAL'S IDENTITY

Note:
- PARSEVAL'S IDENTITY is also called PARSEVAL'S ENERGY THEOREM or RAYLEIGH'S ENERGY THEOREM.
- Energy (For aperiodic) or power (For periodic signals) of a signal in time domain is equal to energy or power in Frequency domain representation.

<u>Note</u>: signal energy is the area under a signal energy density function, which is called energy-density spectrum or energy spectral density

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

<u>Energy-Density Spectrum of $x(t)$:</u>

$$G_x(\omega) = |X(\omega)|^2$$

- It indicates the distribution of signal energy as a function of frequency.

(12) Area under $x(t)$ from $X(\omega)$:

By definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$X(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} x(t) \cdot 1 \cdot dt$$

or $$\boxed{\int_{-\infty}^{\infty} x(t) \cdot dt = X(0)}$$

(13) Area under $X(\omega)$ from $x(t)$

By definition

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \cdot dt$$

$$x(t)\Big|_{t=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{0} \cdot dt$$

or $$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot dt = x(0)}$$

or $$\boxed{\int_{-\infty}^{\infty} X(\omega)\, dt = 2\pi \cdot x(0)}$$

(14) **DUALITY:** (SYMMETRY PROPERTY) or (SIMILARITY THEOREM)

If $\quad x(t) \longleftrightarrow X(\omega)$

then $\quad X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Proof: By definition

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging the variables 't' and ω yields

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi\, x(-\omega) = FT\left[X(t)\right]$$

or $$\boxed{X(t) \longleftrightarrow 2\pi\, x(-\omega)}$$

Note: for even fuctions, $x(-\omega) = x(\omega)$

Hence $\quad X(t) \longleftrightarrow 2\pi\, x(\omega)$.

- Ex: $\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(\omega)$

  $\mathrm{sinc}(t) \longleftrightarrow \cancel{2\pi}\,\mathrm{rect}\,\cancel{(-t\omega)}$

  $2\pi\,\mathrm{rect}(-\omega)$

(15) MODULATION PROPERTY :

If a signal $x(t)$ is multiplied by $\cos\omega_c t$, its spectrum gets translated up and down in frequency by ω_c

$$\text{If} \quad x(t) \longleftrightarrow X(\omega)$$

$$\text{then} \quad x(t)\cos\omega_c t \longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$$

proof:

$$FT\left[x(t)\cos\omega_c t\right] = FT\left[x(t)\cdot\frac{e^{j\omega_c t}+e^{-j\omega_c t}}{2}\right]$$

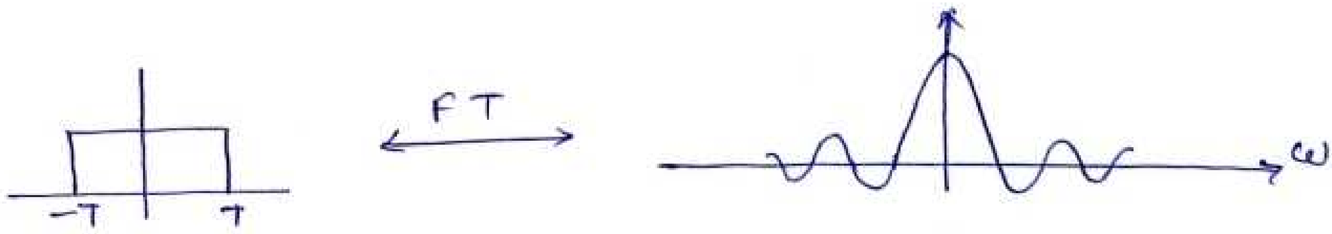
$$= \underbrace{\frac{1}{2}FT\left[x(t)e^{j\omega_c t}\right]}_{\text{from frequency shift property}} + \frac{1}{2}FT\left[x(t)e^{-j\omega_c t}\right]$$

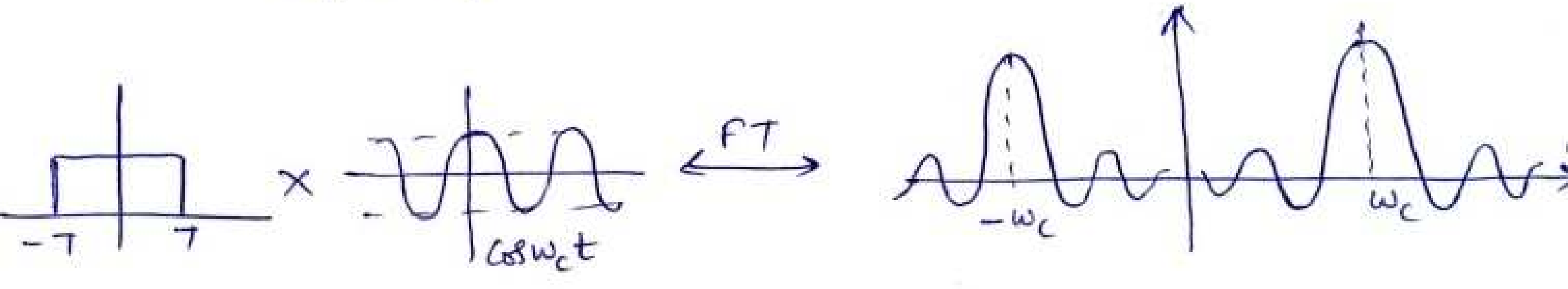
$$= X(\omega-\omega_c)$$

$$\text{Hence} \quad FT\left[x(t)\cos\omega_c t\right] = \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$$

$$\text{Similarly} \quad FT\left[x(t)\sin\omega_c t\right] = \frac{1}{2j}\left[X(\omega-\omega_c)-X(\omega+\omega_c)\right]$$

Ex :





Modulation theorem is the basis of transmission of amplitude-modulated radio broadcasts. When a low frequency audio signal $x(t)$ is multiplied by a radio-frequency carrier wave, the spectrum of audio message is shifted to radio portion of the em spectrum for transmission by an antenna.